Tutorial 2

Resilient Communication Systems
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Preliminaries in Linear Algebra (II)

Hints and Definitions

Assume that $\mathbf{A}^{M\times N}$ and $\mathbf{B}^{N\times N}$ are complex matrices and $a_{m,n}$ and $b_{m,n}$ are entries on the mth row and nth coloum of $\mathbf{A}$ and $\mathbf{B}$ respectively.

a) Frobenius norm:

$$||\mathbf{A}||_F^2 = \sum_{m=1}^{M}\sum_{n=1}^{N} |a_{m,n}|^2,$$

b) Kronecker product:

$$\mathbf{A}\otimes\mathbf{B} = \begin{bmatrix} a_{1,1}\mathbf{B} & \dots & a_{1,N}\mathbf{B} \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ a_{M,1}\mathbf{B} & \dots & a_{M,N}\mathbf{B} \end{bmatrix}$$

c) Hadamard's inequality ($|\mathbf{B}|$ is determinant of the matrix $\mathbf{B}$ and $\mathbf{b}_n$ is nth column of the matrix $\mathbf{B}$):

$$|\mathbf{B}| \le \prod_{n=1}^{N} ||\mathbf{b}_n||$$

d) $\lambda_n(\mathbf{B})$ and $\mathbf{e}_n(\mathbf{B})$ are nth eigenvalue and eigenvector of matrix $\mathbf{B}$ respectively.

Problem 1

Prove the following statements in general or find a counterexample.

1. Is the Frobenius norm of a matrix, a norm on $\mathbb{C}^{N\times M}$?
2. Hadamard's inequality. (When does the equality hold?)
3. Is Kronecker product in general commutative? ($\mathbf{A}\bigotimes\mathbf{B} = \mathbf{B}\bigotimes\mathbf{A}$?)
4. Complex matrix differentiation:

$$\frac{\partial \mathrm{tr}\{\mathbf{AB}\}}{\partial \mathbf{B}^\mathsf{T}} = \mathbf{A}, \qquad \frac{\partial \mathrm{tr}\{\mathbf{AB}^\mathsf{H}\}}{\partial \mathbf{B}^\mathsf{T}} = \mathbf{0}$$

5. Trace-eigenvalue relation:

$$\mathrm{tr}(\mathbf{B}) = \sum_{n=1}^{N} \lambda_n(\mathbf{B})$$

6. Determinant-eigenvalue relation:

$$|\mathbf{B}| = \prod_{n=1}^{N} \lambda_n(\mathbf{B})$$

7. Matrix inversion lemma for $\mathbf{U} \in \mathbb{C}^{N\times M}$ and $\mathbf{V} \in \mathbb{C}^{M\times N}$:

$$(\mathbf{I}_N + \mathbf{U}\mathbf{V})^{-1} = \mathbf{I}_N - \mathbf{U}(\mathbf{I}_M + \mathbf{V}\mathbf{U})^{-1}\mathbf{V}$$

Problem 2

Consider a simple linear system where the vector $\mathbf{x} \in \mathbb{C}^M$ represents the parameters to be estimated, and the measurements $\mathbf{y} \in \mathbb{C}^N$ are collected according to the following linear model:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n},$$

where $\mathbf{H} \in \mathbb{C}^{N\times M}$ is a known matrix that represents the system's channel and $\mathbf{n} \in \mathbb{C}^N$ is the measurement noise. We assume that our estimate of $\mathbf{x}$ is given by:

$$\hat{\mathbf{x}} = \mathbf{W}^{\mathsf{H}}\mathbf{y},$$

where $\mathbf{W}$ is a variable matrix we need to determine.

Our objective is to minimize the expected squared error between the estimate $\hat{\mathbf{x}}$ and the true parameter vector $\mathbf{x}$. This leads to the following optimization problem:

$$\mathbf{W}_{\text{Optimal}} = \underset{\mathbf{W}\in\mathbb{C}^{N\times M}}{\arg\min}\, \mathbb{E}\{\|\mathbf{W}^{\mathsf{H}}\mathbf{y} - \mathbf{x}\|^2\}.$$

In addition, we know the signal-to-noise ratio (SNR) is characterized by:

$$\frac{\mathbb{E}\{\|\mathbf{x}\|^2\}}{\mathbb{E}\{\|\mathbf{n}\|^2\}} = \gamma.$$

Using this information, derive the optimal matrix $\mathbf{W}$ in terms of γ and the known matrix $\mathbf{H}$.